In Search of Mathematical Identity

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Abstract: *Here we show that the common perception of the field of mathematics is not congruent to what the field really is. Dictionary definitions of the word 'mathematics' are taken as indicative of the popular understanding. To discover the true meaning, a questionnaire was circulated around the world (to 7705 individuals and 2339 institutions) and answered by 247 professional mathematicians from 37 countries. All gave a definition of mathematics and the picture emerging from the results is quite radically different from that of dictionaries. The proffered definitions are discussed and a digested definition given at the end of this paper for future consideration. We believe that for the first time a new definition was created based on a democratic poll of a large number of professionals in the relevant field.*

Dozens of dictionaries (in English, German and French) were consulted in order to get an idea of what a standard definition of 'mathematics' would be. Not surprisingly, nearly all dictionaries agreed with each other on the basic ingredients of the field of mathematics. A brief and representative definition is given by the Concise Oxford Dictionary [1]: "The abstract science of number, quantity, and space studied in its own right (pure mathematics), or as applied to other disciplines such as physics, engineering, etc. (applied mathematics)". Many of the even briefer dictionaries merely say 'the science of numbers'. The German dictionary Brockhaus [2] gives a more extensive definition (translation due to the present author): "The science that originated in the practical problems of calculating and measuring and now concerns itself with the interconnections between quantities and patterns and studies their relationships. This conception was enlarged and deepened by mathematical basic research, i.e. through mathematical logic and set theory. As the science of the structures of interrelations, mathematics gives an overview over all possible, purely logical conclusions given a set of basic assumptions (axioms) where any concepts appearing therein define themselves through the axioms". One of the questioned persons gave a definition of a different flavor but in the same spirit: "An abstract philosophical language founded on Propositional Logic, based on the notion of 'Mathematical Proof'. Its generic aim is the discovery and demonstration of propositions obtained by combining axioms, definitions and previously proved theorems". (All anonymous quotations are due to respondents to the questionnaire.)

This is the status quo in terms of what is presented to the public. Note that this view is confirmed by the experiences of the public as given by common school mathematics education (the contents of the English A-level, German Abitur and French Baccalaureate). People are confronted essentially with a numerical and geometric discipline built logically from axioms. That this does not at all reflect the spirit, history, current evolution or researchers of the discipline will become apparent in this article. The fact that the presentation (a textbook) of a body of knowledge does not do justice to the historical and cultural development of that body is a common observation and it is unfortunate as the novice obtains a distorted picture of reality. Indeed, the picture presented above is the ancient Greek view of mathematics as Logic, Geometry and Number Theory and not the current, vastly expanded notion that we treat here.

Mathematics is a large field with many aspects; numbers are just a small portion and logical development merely one feature of the field. Many would say that the study of patterns or structures is more fundamental than either of these (23.6% had this view). They said that mathematics is "a search for patterns and order in the chaos of life" or, "the science of structures and

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patterns that bring order and simplicity to our thinking". One may be satisfied with this definition until we note that saying 'x is the study of y' begs the question after a definition of y. As every word is defined in a dictionary in terms of other words in the same language, one must intuitively understand some words in order to learn more and thus escape the inherent circularity of language. It was suggested (by two people) that 'mathematics' should be such a primitive and undefined word.

According to the Concise Oxford Dictionary, a pattern is "a regular or logical form, order, or arrangement of parts". In this way, we attain to the idea that mathematics expresses something which is amenable to meaningful description: A description which highlights features without having to list the details, which draws attention to important general aspects in a sea of information. The concept of regularity giving rise to predictions is immediately visible in what we choose to call theorems. Given a general law and some pieces of relevant information allows us to draw a number of conclusions without further experimentation. Constructing the general law, this so-called 'inductive step', is the secret behind the applicability of mathematics to natural science.

Instead of patterns, one may say that mathematics is a language; indeed an "unambiguous conversation" (9.8% thought so). As mathematics is also the study of patterns, it is a regular logical language with structure. We consider it to be a language that is almost inherent in the universe (Plato's view) and that we are somehow compelled to learn and develop. It is this assumption that leads us to attempt to communicate internationally based on mathematical principles. A language such as English is much more haphazard having been developed by cultural forces of many interacting individuals over millennia. Being a language we can study the language in its own right (pure mathematics) or study what the language expresses or speaks about (applied mathematics). Indeed "mathematics is a unique language, in which most of the well-studied and best understood human knowledge can be expressed, stored and communicated with the least amount of loss of information. The reason why it functions seems to be its unequalled simplicity!". It is also "the way of thinking" and consists of "eternal truths that can be communicated over generations".

Whatever else it may be, mathematics is structured logically (24.1% felt this was the most important characteristic). One might say that "within mathematics you can include everything that depends entirely and only on logical reasoning" and mathematics is the "study of well-defined things" as well as "the compulsion to define every problem very precisely".

The opinion that mathematics is numerical is widespread (17.2% saw this as the defining property). Certain sub-fields such as topology easily show that mathematics is not totally numerical. We may say that mathematics is the "science about numbers and everything that derives from there" or simply "the study of numbers", or "the science of numbers -mathematics explores every aspect of numbers, how we use them, how they are applied to day-to-day life and how they are applied to science". To go so far as to say mathematics is "the pursuit of happiness through numbers" is taking us to the next great aspect of the discipline.

There is an eternal debate over whether mathematics is discovered or created. No matter which way the individual chooses to follow, mathematics is a creative pursuit (4.6% felt this crucial). This artistic aspect cannot be ignored in defining the field but it is entirely ignored in most textbooks on the subject. Mathematics is "the science of the infinite", "nature's manual" and "combinatorial intellectual art". One does not need to be "strongly opposed to all flavours of Platonism and side with a version of formalism with certain pragmatic overtones". Some quote Weierstrass: "A mathematician who is not also a little bit of a poet is not a good mathematician". Whatever it is that we are doing, we do it "for the fun of it". We seek "purest beauty in a human-defined universe of logical structures" and we "feel 'mathematics' as a mental outburst of an animate being". Mathematics is "just as difficult as defining 'Brahman': not like this, not like this" and it is also "the will for uniqueness". Mathematicians also share the artistic spirit of "curiosity, inductive reasoning, intuition, [and] logic or deductive reasoning".

Clearly mathematics is useful. As Galileo said: "Mathematics is the alphabet in which God described the universe". The natural sciences are concerned with predicting the future state of physical systems given their present state. As a matter of practicality, we would like this prediction in numerical form and arrived at using a logical, structured but also creative method that has a degree of generality and universality. As such, mathematics becomes the "queen of all sciences" (13.2% view it as such). In the spirit of wanting to accomplish physical aims, mathematics becomes "the fine art of problem solving" and "the way to model life". Some degrade it to the level of a mere tool and let it be "a means to teaching physics" for the reason that "everything, sooner or later, comes down to measurement".

Mathematics may, in fact, be "that which mathematicians do" (4.6% expressed this view). They build "huge buildings, some beautiful, some useful, resting safely on tiny foundations and with almost zero costs" and follow "exact rules to the utmost accuracy and all that with brilliant imagination" so that "the process of mathematics is the organised investigation of patterns derived from axioms; the body of mathematics is the collected remembrance of those investigations". Choosing to define a field by the activities of its researchers is a convenient way to side-step the difficulty and it is mirrored by the attempt to merely point out the difficulty (as 1.7% have done): "Mathematics has a fractal structure: If you look closer and closer to its subjects, more and more structures evolve.". It is hard to tell what is mathematics and what is not and thus to define the field. We may say that "everything is mathematics, it's just that people aren't aware of it".

Having obtained the opinion of 247 professionals and the distribution of their answers we digest them: 24.1% logic, 23.6% patterns, 17.2% numbers, 13.2% modelling, 9.8% language, 4.6% art, 4.6% researchers, 1.7% fuzziness and 1.1% dictionaries. Clearly mathematics studies patterns in a logical, numerical way, develops itself by mean of creative researchers and can be used to talk about the world and to assist natural science thereby being hard to define. It is "one of the great achievements of mankind" and one might say that "the definition would not contain the word 'number', and it would say something about the artistic aspect of math".

Using the information gained, we wish to use it all to obtain a new definition of the field that can be used in future dictionaries to more accurately represent the discipline. Before this, however, we reflect on the fact that the creative, language and evolutionary sides (together 19.0%) are usually ignored and the applied aspect (13.2%) downplayed in school mathematics which is the last most people see of the field. While teaching mathematics, the author believes, it is important to give a feel for the structure of the whole of mathematics, for the creativity and artistry necessary to produce the so surgically presented results and for the applicability of mathematics to a wide variety of natural as well as social phenomena. This will heighten the appeal of the field and create a different vision among the public who, by and large, seem to view mathematics as principally 'difficult' as well as essentially numerical, logical and abstract. Having allowed myself a little moralizing, here is the digested definition of mathematics that is presented for consideration:

Math.e.mat.ics n. pl. [from the Greek mathema "to learn" which came from the Sanskrit medha "wisdom and intelligence"]. A collection of subjects that investigate particular kinds of patterns that are derived from the physical universe, abstract thought or the imagination with the aim of giving logically correct analyses of the properties of these patterns. Examples of the subjects that make up mathematics are: Logic (studies the process of obtaining conclusions abstractly), Set Theory (analyses the relationships between abstract objects), Geometry (deals with points and lines in any kind of space), Topology (considers shape of objects in space irrespective of their size), Algebra (examines the relationships of symbols given certain assumed properties of these symbols), Analysis (investigates functions and continuity) and Number Theory (studies the abstract properties of numbers).

Mathematics is typically divided into pure and applied mathematics, a distinction that applies to the practise and motivation rather than to the subject. If mathematics is practised with a benefit to subjects outside of mathematics in mind (natural sciences, social studies, industry, humanities etc.), then it is applied mathematics; if it is practised for its own sake, it is pure mathematics. While the record of the constituting subjects are in the form of short statements of the properties of the investigated patterns (theorems) and the logically consistent justification of these statements (proofs) the process of arriving at this linear, logical and scientific record is significantly different. The practitioner of mathematics (mathematician) proceeds by using a keen sense of beauty and elegance of the pattern to be studied and intuition about what properties the pattern could have and how one might prove it. As an evolving domain of learning, mathematics is thus as much an artistic discipline as a science. Mathematics is also a language in which any kind of regularity can be conveniently expressed and, for that reason, it is the language of natural science and engineering and quickly becoming an indispensable part of the language of social, economic and other fields of study. The collection is so varied and vast that it is frequently uncertain whether a given factoid is to be included under the umbrella that is mathematics.

References

[1] *The Concise Oxford Dictionary*, ninth edition (Oxford University Press, Oxford, 1997).
[2] *Der Neue Reader's Digest Brockhaus*, second edition (Verlag Das Beste, Stuttgart, 1973).